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Non-local equations for general relativity

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The field equations for two non-local variables, equivalent to the Einstein vacuum equations, are presented. These variables are the holonomy operator associated with special paths and the light cone cut function.

Starting from these equations, one shows via a perturbation argument that a single, fourth-order equation for the cut function can be derived. This single equation encodes the entire conformal structure of a vacuum space-time. The same perturbation technique yields, via quadratures, solutions to the vacuum Einstein equations to any order.

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1. Introduction

Among the many contributions that Roger Penrose has made to the field of General Relativity, the use of non-locality and self-duality have played a major, though probably not fully appreciated, role in the development of the field [1]. Ashtekar's formulation of canonical gravity [2] and subsequent developments are based in an essential way on the idea of self-duality while the theory of the non-linear graviton, H-space [3] and the present work are based on both non-locality and self-duality. The work reported here thus owes a major debt to Roger's ideas and encouragement. We take this opportunity to thank Roger for the years of collaboration and deep friendship.

In this reformulation of General Relativity [4] (a generalization of H-space ideas) two non-local variables, the holonomy operator associated with specific closed curves (whose definition is based on duality) and the light cone cut function of null infinity are used as the basic variables. They replace the usual metric and connection of the space–time, which become derived concepts.

A purpose of this note is, first of all, to present an informal review of this approach, trying to avoid technical details. We also wish to present some further developments and a new point of view towards this work.

In section 2 we review the main results of our reformulation, giving, in symbolic form, the coupled non-local field equations for the holonomy operator and cut function. (An outline of the derivation of these equations is given in appendix A while the full set of equations is given in appendix B.) We then raise the question whether it is possible, by increasing the differential order of the equations, to eliminate the holonomy operator from the equations and obtain *a single higher-order equation just for the cut function* in terms of the free data, which would be equivalent to the (conformal) Einstein equations [5].

In section 3 a perturbation procedure to obtain this equation (which will be referred to as the Light Cone Cut Equation, LCCE) is discussed. In the linear approximation, the resulting LCCE is found and shown to be the same as an equation suggested by Mason [6]. The LCCE is also extended to second order in the perturbation expansion where we can identify the interaction between the self-dual and anti-self-dual parts of the free data. Finally, we argue from the perturbation expansion that the full LCCE in principle does exist.

In section 4 we use the same perturbation expansion to argue that the solution space of the LCCE is four dimensional, i.e., that the LCCE, an equation for local cross-sections of a line bundle over the sphere, defines via the space of solutions a four-dimensional manifold, the space–time itself. From these local cross-sections a (conformal) metric on this manifold can be obtained.

2. The non-local field equations

We first give a brief discussion of our earlier work on encoding the conformal properties of an Asymptotically Simple Space–Time (ASST) in terms of a single non-local function, the light cone cut function [7].

We begin with an ASST, the physical manifold M and its conformal completion, i.e., M plus the conformal boundary $\mathcal{I}^+ \bigcup \mathcal{I}^-$. Though the radiation data for the ASST could be chosen on either $\mathcal{I}^+$ or $\mathcal{I}^-$, for simplicity we will make the definite choice $\mathcal{I}^+$.

On $\mathcal{I}^+$ we choose the usual "Bondi coordinates", $(u, \zeta, \bar{\zeta})$, where u labels the Bondi slicing of $\mathcal{I}^+$ and the complex stereographic coordinates $(\zeta, \bar{\zeta})$ label the sphere of null generators. Any two-surface on $\mathcal{I}^+$ is given by some function $u = f(\zeta, \bar{\zeta})$ and will be referred to as a "cut" of $\mathcal{I}^+$ or simply as a "cut". We now consider any interior point of M, with local coordinates x^a, and construct its null cone N_x. The intersection of N_x with $\mathcal{I}^+$ is a preferred cut, C_x, which we refer to as a "light cone cut". It will be locally described by a function, the "light cone cut function", associated with the point x^a, i.e.,

$$u = Z(x^a, \zeta, \bar{\zeta}). \tag{1}$$

The function Z plays a dominant and critical role for us. First of all it allows us to "find" the points on $\mathcal{I}^+$ that are connected, via null geodesics, to x^a. Second, it is the knowledge of the cut or cut function that determines the apex, x^a, of the light cone, i.e. the light cone cuts determine the space–time points. Most important is the fact that knowledge of Z is equivalent to knowledge of the conformal structure of M. This can be seen from the following argument: Note that $u = Z(x^a, \zeta, \bar{\zeta})$ has two meanings: the one just given, $C_x = N_x \bigcap \mathcal{I}^+$, i.e. the light cone cuts, and the second, which arises from holding $(u, \zeta, \bar{\zeta})$ constant but varying the x^a, yielding a characteristic surface, i.e. all points x^a that get to $(u, \zeta, \bar{\zeta})$ on $\mathcal{I}^+$ via null geodesics. This surface is the past cone of the point $(u, \zeta, \bar{\zeta})$. Taking the gradient of Z at a fixed point, x^a, we obtain, by definition, a null covector

$$\ell_a = \nabla_a Z(x^b, \zeta, \bar{\zeta}).$$

By letting $(\zeta, \bar{\zeta})$ range over the sphere, the null covector ranges over the cone of null "directions" at x^a, thus yielding the conformal structure. From Z, one can explicitly construct the conformal metric [7].

It had long been our expectation that a simple equation for Z of the form

$$\eth^2 Z = \Lambda(Z, \eth Z, \bar{\eth} Z, \eth\bar{\eth} Z, \zeta, \bar{\zeta}, \text{data}) \tag{2}$$

could be found that would encode the conformal structure of the vacuum Einstein equations. (The characteristic data for general relativity are given by the Bondi shear, σ_{B} , a complex function on $\mathcal{I}^+$.) This expectation was based on the observation that many vacuum space–times could be found from an equation of that form, e.g., the regular solutions of

$$\eth^2 Z = 0 \tag{3}$$

yield the Minkowski space light cone cuts and the regular solutions of the "good cut equation" [3]

$$\eth^2 Z = \sigma_{\mathrm{B}}(Z, \zeta, \bar{\zeta}), \tag{4}$$

with $\sigma_{\mathrm{B}}(u, \zeta, \bar{\zeta})$ the free Bondi data, yield the cuts of H-spaces.

Note that (3) and (4) are equations for the angular behaviour of the function Z; no mention is made of space–time points. The idea is that the solutions are to depend on a four-parameter set, i.e., four constants of integration x^a, which *define* a four-manifold, the space–time itself. The functional form of Z evaluated at the point x^a yields the conformal metric. After considerable effort, we no longer believe that an equation of the form (2) can be found for general vacuum space–times. On the other hand, two alternatives to (2) have arisen.

(i) Lionel Mason has argued, from the vanishing of the Bach tensor [6], that instead of (2) there should be a single fourth-order equation, the Light Cone Cut Equation (LCCE), of the form

$$\eth^2\bar{\eth}^2 Z = F[Z, \text{data}]. \tag{5}$$

This is a generalization of (2), which should encode the conformal vacuum Einstein equations. We emphasize that the form of F is not known at the present time. From general considerations it is, however, surmised that it will be a universal non-local functional of both Z and the characteristic data, the Bondi shear $\sigma_{\mathrm{B}}(u, \zeta, \bar{\zeta})$. Again the solution space of the LCCE is to be a four-parameter set, the space–time manifold itself with the solutions yielding the conformal metric on the manifold. We will return to this issue in section 4.

(ii) A second approach to the generalization of (2) came with the realization [4] that by introducing a set of auxiliary variables (namely the components of the holonomy operator, H, associated with a special set of closed paths) we could write a pair of coupled angular differential equations that encoded the full vacuum Einstein equations, including the conformal scaling. Though in detail they are reasonably complicated (see appendix B), symbolically they have the simple form

$$\eth^2 Z = L[Z, H, \text{data}], \tag{6}$$

$$\eth H = K[Z, H, \text{data}]. \tag{7}$$

The point of view towards (6) and (7) is to be similar to that of (5) in that they are both angular differential equations whose solution space defines the manifold and whose solutions Z and H yield the vacuum Einstein metrics. (Z by itself only yields the conformal metric but with H the scaling is determined.) An immediate question arises: what relationship, if any, is there between (5) and (6), (7)? One would expect, since Z encodes the conformal metric, that by taking two angular derivatives of (6), H could be eliminated via (7) and an equation of the form (5) obtained. This would be an important result since it would yield the explicit form of F. Though we believe that this calculation can be explicitly carried out, we, nevertheless, have not yet succeeded in doing so – the calculations being simply too complicated. However, a perturbative procedure to obtain F is given in section 3. We plan to return to this issue in the future.

3. Perturbations

It is the purpose of this section to take the explicit form, from appendix B, of the symbolic equations (6) and (7), expand them in powers of a small parameter ϵ which measures the deviation from flatness and eliminate H from the equations term by term leaving finally an equation only for Z, i.e. the LCCE. From the assumption that ϵ enters as a multiplicative factor of the Bondi shear, i.e. via $\epsilon\sigma_{\rm B}$, it becomes clear that the expansions have the form

$$Z = Z_0 + \epsilon Z_1 + \epsilon^2 Z_2 + \cdots, \qquad e_a^i = e_{0a}^i + \epsilon e_{1a}^i + \epsilon^2 e_{2a}^i + \cdots, \tag{8}$$

$$H = \epsilon H_1 + \epsilon^2 H_2 + \cdots, \qquad h = \epsilon^2 h_2 + \epsilon^3 h_3 + \cdots. \tag{9}$$

By direct substitution of (8) and (9) into (B.11) we have as the only zeroth-order term

$$\eth^2 Z_{0,a} = 0 \quad \to \quad \eth^2 Z_0 = 0. \tag{10}$$

Applying $\bar{\eth}$ twice yields the zeroth-order LCCE,

$$\bar{\eth}^2\eth^2 Z_0 = 0, \tag{11}$$

whose solution is

$$Z_0 = x^a \ell_{0a}$$

with

$$\ell_{0a}(\zeta,\bar{\zeta}) = \frac{1}{2\sqrt{2}P}(1+\zeta\bar{\zeta}, \zeta+\bar{\zeta}, {\rm i}(\zeta-\bar{\zeta}), -1+\zeta\bar{\zeta}),$$

$$P = \tfrac{1}{2}(1+\zeta\bar{\zeta}).$$

Continuing the expansion in eqs. (B.5)–(B.15), the *first-order* terms yield

$$\begin{aligned}\eth^2 Z_{1,a} = {} & (\dot{\sigma}_{\rm B} + H_{1-}^{(-)} - \eth H_{01}^{(-)}) Z_{0,a} \\ & + 2H_{1-}^{(-)}\eth\bar{\eth} Z_{0,a} - 2H_{01}^{(-)}\eth Z_{0,a} + \eth H_{1-}^{(-)}\bar{\eth} Z_{0,a},\end{aligned} \tag{12}$$

$$\begin{aligned}\bar{\eth} H_{01}^{(-)} &= -H_{0+}^{(-)}, & \bar{\eth} H_{1-}^{(-)} &= 2H_{01}^{(-)}, & \bar{\eth} H_{0+}^{(-)} &= \eth\dot{\bar{\sigma}}_{\rm B}, \\ \eth H_{01}^{(+)} &= -H_{0-}^{(+)}, & \eth H_{1+}^{(+)} &= 2H_{01}^{(+)}, & \eth H_{0-}^{(+)} &= \bar{\eth}\dot{\sigma}_{\rm B}.\end{aligned}$$

By applying $\bar{\eth}^2$ to the first equation and eliminating the H via the second set we obtain

$$\bar{\eth}^2\eth^2 Z_{1,a} = \eth^2(\dot{\bar{\sigma}}_{\rm B} Z_{0,a}) + \bar{\eth}^2(\dot{\sigma}_{\rm B} Z_{0,a}), \tag{13}$$

which immediately integrates to

$$\bar{\eth}^2\eth^2 Z_1 = \eth^2\bar{\sigma}_{\rm B}(Z_0) + \bar{\eth}^2\sigma_{\rm B}(Z_0), \tag{14}$$

the first-order LCCE. Since Z_0 is known and there is a simple Green's function for the operator $\eth^2\bar{\eth}^2$ one can write the solution Z_1 as a sphere integral over the data.

[Note that if (14) is written as

$$\bar{\eth}^2\eth^2 Z = \eth^2\bar{\sigma}_B(Z) + \bar{\eth}^2\sigma_B(Z), \tag{15}$$

and understood to be an equation for Z accurate only to first order, *it is equivalent to the linearized (conformal) Einstein equations.*]

With considerably more effort this expansion procedure can be continued, and the second-order H eliminated, obtaining the LCCE accurate to second order,

$$\begin{aligned}\bar{\eth}^2\eth^2 Z = {}& \eth^2\bar{\sigma}_B(Z) + \bar{\eth}^2\sigma_B(Z) \\ & + \int_r^\infty \Big(\tfrac{3}{4}\eth\bar{\eth}(\Lambda_{,r'''}\bar{\Lambda}_{,r'''}) \\ & + \eth\bar{\eth} \int_{r'''}^\infty \int_{r''}^\infty (\Lambda_{,r'r'}\bar{\Lambda}_{,r'r'})\,\mathrm{d}r'\,\mathrm{d}r'' \Big)\mathrm{d}r'''.\end{aligned} \tag{16}$$

An important observation is that the equation is no longer local. It now depends on radial integrals along the null geodesics from the field point to $\mathcal{I}^+$. This is a manifestation of the non-Huygens behaviour of the non-linear Einstein equations. It is also an example of Penrose's zig-zag integrals [8].

It is easy to see that at the nth order of the approximation, the coefficients of the H_n are exactly the same as the coefficients of the H_{n-1} at the $(n-1)$th order. From this observation it follows that H can be systematically eliminated at all orders. Assuming that the procedure converges, H could, in principle, be eliminated – leaving an exact form for the LCCE.

4. Conclusion

We have adopted the point of view that the (conformal) Einstein equations can be rewritten as a single non-local, non-linear angular equation of the form

$$\eth^2\bar{\eth}^2 Z = F[Z, \text{data}]. \tag{17}$$

where the F is a universal functional of Z and the data. We have expressed the idea that this equation, the LCCE, which at the start contains no mention of space–time itself, nevertheless somehow defines the space–time as the space of solutions. The set of solutions (with some as yet unspecified regularity condition) are to be parametrized by four "constants of integration", which become, by definition, the local coordinates x^a of the space–time manifold. The solutions thus would have the form

$$u = Z(x^a, \zeta, \bar{\zeta}). \tag{18}$$

The question is: why should one expect or hope that the LCCE would have such a property? First of all one had the same situation with the "good cut equation", eq. (4) – it was not clear why it should have a four-parameter set of regular solutions. Basically the reason is that the kernel of the $\eth^2$ operator is four dimensional. This was used first in a perturbative argument for the four dimensionality of the solution space and later to give a rigorous index theorem argument [9].

Basically we have the same situation here. The kernel of $\eth^2\bar{\eth}^2$ is again four dimensional, i.e., Z_0 depends on four parameters, x^a. In an iterative scheme, Z_0 continually gets reinserted as the "driving" term for the higher approximations, always yielding the solutions as functions of the same four parameters, x^a.

This certainly does not constitute a proof of the four dimensionality of the solution space – it is only a plausibility argument. One hopes that with an exact form for F a proof via an index theorem will emerge. It is perhaps worth mentioning that, once the cut function is obtained, i.e., $u = Z(x^a, \zeta, \bar{\zeta})$, the construction of the conformal vacuum metric is a straightforward kinematical procedure [7].

There is an important caveat to the above remarks that must be stated and explained. For the sake of simplicity and clarity we omitted the discussion of a serious structural complication in eq. (17) and its perturbative version eq. (16). One sees that there are both integrals and derivatives with respect to the (as yet undefined) variable r in F. We will now define r and explain its role in the "structural complication".

Assuming that we know

$$u = Z(x^a, \zeta, \bar{\zeta}),$$

we can define $\omega = \eth Z(x^a; \zeta, \bar{\zeta})$, $\bar{\omega} = \bar{\eth} Z(x^a; \zeta, \bar{\zeta})$ and $r = \eth\bar{\eth} Z(x^a; \zeta, \bar{\zeta})$.

These four equations can be inverted (for Minkowski space and spaces close to Minkowski space), yielding

$$x^a = x^a(u, \omega, \bar{\omega}, r; \zeta, \bar{\zeta}).$$

With this relationship, differentiation and integration with respect to r becomes well defined, e.g., $\Lambda_{,r} = \Lambda_{,a}\, \mathrm{d}x^a/\mathrm{d}r$.

The complication is now clear: the functional F depends on both the r derivatives and integrals. These are, however, only defined when the solutions are *known*. Equation (17) is thus, in some sense, not meaningful. On the other hand, if a function $Z(x^a, \zeta, \bar{\zeta})$ exists, it can be tested to see if it satisfies (17). In addition (17) has a perfectly well-defined meaning in a perturbative sense: at the nth order of a perturbation calculation the functional F depends on the already known $n-1$ terms, all of which are known functions of r. Though we do not fully understand eq. (17), it appears to have a meaning in a "bootstrap" sense; the solutions Z define the x^a and hence r, while the r is used to define eq. (17) itself.

Appendix A

In this appendix we give a brief review of a slightly unconventional way of rewriting the Einstein equations [10] and then use these equations as a first step in deriving field equations for the holonomy operator.

We start with the ordinary Yang–Mills equations on an unspecified Lorentzian manifold, for the O(3,1) gauge group. In the vector representation, the connection $\gamma_{a\,j}^{\;i}$ will have (in addition to the space–time index a) a pair of Lorentzian internal indices, i, j, which can be raised and lowered with the (internal) Minkowski metric so that the connection will be antisymmetric in these internal indices and hence can be decomposed into a self- and anti-self-dual pair, i.e.,

$$\gamma_a{}^{ij} = \gamma_a^{(+)ij} + \gamma_a^{(-)ij}, \tag{A.1}$$

where self- and anti-self-dual are defined by

$$\gamma_a^{(\pm)ij} = \gamma_a{}^{ij} \mp \gamma_a^{*ij} \tag{A.2}$$

and duality by

$$\gamma_a^{*ij} = \tfrac{1}{2}\epsilon^{ij}{}_{kl}\gamma_a^{kl}, \tag{A.3}$$

where ϵ_{ijkl} is the alternating symbol with $\epsilon_{0123} = -1$.

The curvature tensor, the Bianchi identities and the Yang–Mills field equations then also decompose into the (internal space) self- and anti-self-dual parts, i.e., there is no coupling between the self- and anti-self-dual parts. One has (suppressing the internal indices)

$$F_{ab} = F_{ab}^{+} + F_{ab}^{-},$$

where the self- and anti-self-dual curvatures are constructed from the self- and anti-self-dual connections. The Bianchi identities become

$$\nabla_{[e}F_{ab]}{}^{\pm} + [\gamma_{[e}^{\pm}, F_{ab]}^{\pm}] = 0, \tag{A.4}$$

and the Yang–Mills field equations are

$$\nabla^a F^{\pm}_{ab} + [\gamma^{\pm a}, F^{\pm}_{ab}] = 0. \tag{A.5}$$

One is thus dealing with two independent Yang–Mills connections and fields. It is possible to further decompose each of the two curvature tensors now on the space–time indices, into its space–time, self- and anti-self-dual parts, where we have used the existence of the Lorentzian metric. We will refer to space–time dual statements as left dual and internal dual statements as right dual. The full curvature then has four terms,

(i) the left and right self-dual part, ${}^+F^+_{ab}$;

(ii) the left anti-self-dual and right self-dual part, ${}^-F^+_{ab}$;

(iii) the left self-dual and right anti-self-dual part, ${}^+F^-_{ab}$;

(iv) the left anti-self-dual and right anti-self-dual part, ${}^-F^-_{ab}$.

Parts (1) and (2) are coupled as are parts (3) and (4), in the sense that they depend, respectively, on the γ^+ and γ^-. If we now make the *algebraic assumption* that the curvature parts, (2) and (3), both vanish, i.e.,

$${}^-F^+_{ab} = 0 \quad \text{and} \quad {}^+F^-_{ab} = 0, \tag{A.6}$$

then we are left with two curvatures, ${}^+F^+_{ab}$ and ${}^-F^-_{ab}$ curvatures, coming, respectively, from the two independent connections. This algebraic assumption has automatically imposed the Yang–Mills field equations on the connection. The field equations in each case are identical to the Bianchi identities, i.e., eq. (A.5) follows from (A.4) after dualing. We are thus dealing with two Yang–Mills fields, a (left) self- and a (left) anti-self-dual Yang–Mills field.

To this system we now add another variable, namely a space–time orthonormal tetrad λ^i_a, compatible with the unknown Lorentzian metric, i.e., a soldering form to be used to connect the space–time indices with the internal indices, e.g., $A_i\lambda^i_a = A_a$. The role of λ^i_a will be to connect or relate the two (originally) independent connections γ^+ and γ^- to the space–time geometry. This relationship is given by the Cartan structure equation

$$\nabla_{[a}\lambda^i{}_{b]} = (\gamma^{(+)i}{}_{j[a} + \gamma^{(-)i}{}_{j[a})\lambda^i{}_{b]}\,. \tag{A.7}$$

Equations (A.6) and (A.7) are equivalent to the vacuum Einstein equations with cosmological constant [10].

We now introduce the holonomy operator associated with the SO(3,1) connection γ_a introduced in eq. (A.1). In symbolic form this operator is defined as

$$H = \mathrm{P}\exp\left(\oint_C \gamma_a \mathrm{d}x^a\right),$$

where C is an arbitrary closed path with an initial point in space–time and P exp denotes the path ordered exponential.

Note that this operator is defined on the space of loops, an infinite-dimensional space. We will, however, restrict ourselves to two special six-dimensional subspaces of loop space. Essentially these loops are the infinitesimally narrow, long triangles with apex at x^a, bounded by two neighbouring null geodesics and connected by the geodesic deviation vector at $\mathcal{I}^+$.

More precisely, we denote by $\ell_x(\zeta,\bar{\zeta})$ the null geodesic that starts at x^a and ends at the $(\zeta,\bar{\zeta})$ generator of $\mathcal{I}^+$. We introduce two types of paths, defined as the infinitesimal triangles $\Delta_x(\zeta,\bar{\zeta})$ [and $\overline{\Delta}_x(\zeta,\bar{\zeta})$] formed by two neighbouring geodesics $\ell_x(\zeta,\bar{\zeta})$ and $\ell_x(\zeta + \mathrm{d}\zeta,\bar{\zeta})$ [and $\ell_x(\zeta,\bar{\zeta})$ and $\ell_x(\zeta,\bar{\zeta}+\mathrm{d}\bar{\zeta})$] and connected at I^+ with the connecting vectors M^a [and $\overline{M}^a$] on I^+. Of course Δ_x and $\overline{\Delta}_x$ both lie on the null cone C_x. Since these paths are very narrow, the holonomy operators associated with them will be the identity operator plus a correction term for each of them. It is the correction terms that are referred to as the differential holonomy operators associated with the paths $\Delta_x(\zeta,\bar{\zeta})$, and $\overline{\Delta}_x(\zeta,\bar{\zeta})$. We denote them by

$$H(x^a,\zeta,\bar{\zeta})\,\mathrm{d}\zeta\ ,\ \overline{H}(x^a,\zeta,\bar{\zeta})\,\mathrm{d}\bar{\zeta}. \tag{A.8}$$

A second basic variable, the light cone cut function $Z(x^a,\zeta,\bar{\zeta})$ (see section 2) contains or codes the conformal information of the underlying conformal structure. It is defined as the intersection of the light cone emanating from an interior point x^μ and I^+, the future null boundary attached to an ASST.

One can show [7] that all the components of the conformal metric are explicit functions of

$$\Lambda(x^a,\zeta,\bar{\zeta}) \equiv \eth^2 Z. \tag{A.9}$$

As we will see below, Z and H must be coupled if the holonomy operator is associated with the space–time metric connection. However, at this point it is convenient to think that H is associated with an independent SO(3,1) Yang–Mills connection whereas $Z(x^a,\zeta,\bar{\zeta})$ is assumed to be a known function that describes the conformal gravitational background.

One shows [4], using a non-abelian version of Stoke's theorem, that H is related to the Yang–Mills curvature tensor in the following way,

$$H = \int_{s_0}^{\infty} (F^+_{ab} + F^-_{ab})\,\ell^a M^b\,\mathrm{d}s = h^{(+)} + H^{(-)}, \tag{A.10}$$

$$\overline{H} = \int_{s_0}^{\infty} (F^+_{ab} + F^-_{ab})\,\ell^a \overline{M}^b\,\mathrm{d}s = H^{(+)} + h^{(-)}, \tag{A.11}$$

where H and h with the plus and minus signs denote the self-dual and anti-self-dual parts of H and $\overline{H}$, and are defined in the obvious manner from the integrals of the F's with the plus and minus signs.

One can invert these equations and reexpress F_{ab}^{+} and F_{ab}^{-} in terms of H and $\overline{H}$ [4]. This relation is symbolically written as

$$F_{ab}^{-} = F_{ab}^{-}(H^{(-)}, h^{(-)}) \tag{A.12}$$

with an analogous equation for F_{ab}^{+}. If we now impose eq. (A.6), i.e., ${}^{+}F_{ab}^{-} = 0$, on eq. (A.12), then $H^{(-)}$ and $h^{(-)}$ are no longer independent variables. They become related by [4]

$${}^{+}F_{ab}^{-}(H^{(-)}, h^{(-)}) = 0, \tag{A.13}$$

or in detail,

$$[q^{-1}h_{,r}^{(-)}]_{,r} + \delta[q^{-1}\Lambda_{,r}h_{,r}^{(-)}]_{,r} = [q^{-1}\overline{\Lambda}_{,r}H_{,r}^{(-)}]_{,r} + \delta[q^{-1}H_{,r}^{(-)}]_{,r}\,, \tag{A.14}$$

where r denotes a radial parameter defined by Z on the null geodesic, $\delta = (\sqrt{q}-1)/\Lambda_{,r}$, $q = 1 - \Lambda_{,r}\overline{\Lambda}_{,r}$, and $\Lambda(x,\zeta,\overline{\zeta})$ is the function introduced in eq. (A.9). Note that all these quantities are obtained from the assumed known Z.

One can explicitly solve this equation for $h^{(-)}$ by quadratures and write the solution as $h^{(-)} = J[H^{(-)}]$, a linear functional of $H^{(-)}$. In this sense, we consider $h^{(-)}$ as derived from $H^{(-)}$, with $H^{(-)}$ our basic variable.

Finally, we address the issue of obtaining field equations for $H^{(-)}$. The idea is to integrate the Bianchi identities, which become field equations by virtue of (A.6), on the infinitesimally narrow but infinitely long volume ΔV bounded by a cap on $\mathcal{I}^{+}$ and the triangular regions $\Delta_x(\zeta,\overline{\zeta})$, $\Delta_x(\zeta,\overline{\zeta}+\mathrm{d}\overline{\zeta})$, $\overline{\Delta}_x(\zeta,\overline{\zeta})$ and $\overline{\Delta}_x(\zeta+\mathrm{d}\zeta,\overline{\zeta})$ (i.e., a pyramid-like figure with apex at x^a and base at infinity). Integrating (A.4) over ΔV, we obtain [4]

$$\overline{\eth}H^{(-)} - \eth(h^{(-)} - \overline{A}_{\mathrm{R}}^{(-)}) + [H^{(-)} - A_{\mathrm{R}}^{(-)}, h^{(-)} - \overline{A}_{\mathrm{R}}^{(-)}] = 0, \tag{A.15}$$

with

$$A_{\mathrm{R}}^{(-)} = A^{(-)}(Z,\zeta,\overline{\zeta})$$

and $A^{(-)}(u,\zeta,\overline{\zeta})$ the "free data" at $\mathcal{I}^{+}$. That is, A_{R} is the restriction of the Yang–Mills data to the light cone cut. The idea then is to find solutions of this equation that are regular on the cut. The non-Huygens nature of the original field equations is explicitly exhibited in $h^{(-)}$, which is an integral functional of $H^{(-)}$. Equation (A.15) is equivalent to the anti-self-dual Yang–Mills equations for an O(3,1) gauge group on an asymptotically simple background given by $Z(x,\zeta,\overline{\zeta})$.

We *now* assume that H is the holonomy operator associated with the metric connection of the ASST and that the metric of the space–time satisfies the vacuum equations. It is well known [11] that for ASST the spin coefficients have a very simple asymptotic form, with only one complex degree of freedom.

Thus, the free data A, which are an asymptotic component of the connection γ, assume a very restrictive form, given by $A = A^{(+)} + A^{(-)}$ with

$$A_{ij}^{(+)} = \begin{pmatrix} 0 & 0 & 0 & -\dot{\sigma}_{\mathrm{B}} \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \dot{\sigma}_{\mathrm{B}} & 0 & 0 & 0 \end{pmatrix}, \tag{A.16}$$

and

$$A_{ij}^{(-)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{A.17}$$

with σ_{B} the gravitational data [4]. On the other hand, these equations are clearly incomplete since the cut function that would enter into the equations for H should be obtained from knowledge of the very metric we are trying to solve for.

The idea then is to relate the two variables H and Z. This relationship completes our set of equations equivalent to the vacuum Einstein equations.

A sketch of the derivation of this relationship begins with the equations [4]

$$\eth e_a^k = (H^k{}_j - A^k{}_j)e_a^j, \qquad \bar{\eth} e_a^k = (\overline{H}^k{}_j - \overline{A}^k{}_j)e_a^j, \tag{A.18}$$

a requirement that the null space-time tetrad e_a^k is parallel propagated from $\mathcal{I}^+$ to the field point x^a along a null geodesic $\ell_x(\zeta,\bar{\zeta})$ using the Yang-Mills connection $\gamma^+ + \gamma^-$. The next step is to relate this tetrad to the natural gradient basis

$$(Z_{,a}, \eth\bar{\eth}Z_{,a}, \eth Z_{,a}, \bar{\eth}Z_{,a}).$$

This is done by choosing one of the "legs" as

$$e_a^0 = \ell_a = Z_{,a}. \tag{A.19}$$

It immediately follows from eq. (A.16) that

$$\eth Z_{,a} = m_a + H^0{}_i e_a^i = m_a - H_{01}\ell_a + H_{1+}m_a + H_{1-}\bar{m}_a, \tag{A.20}$$

and

$$\bar{\eth} Z_{,a} = \bar{m}_a + \overline{H}^0{}_i e_a^i = \bar{m}_a - \overline{H}_{01}\ell_a + \overline{H}_{1+}m_a + \overline{H}_{1-}\bar{m}_a. \tag{A.21}$$

Taking $\bar{\eth}$ of (A.20) or $\eth$ of (A.21) and using (A.18) yields

$$\eth\bar{\eth}Z_{,a} = n_a - \ell_a + (\bar{\eth}H_{1i} - \overline{H}_{-i} + H_{1k}(\overline{H} - \overline{A})^k{}_i)e_a^i. \tag{A.22}$$

Equations (A.19)–(A.22) are the needed relationship between the gradient basis and the null tetrad. This allows us to go back and forth between these two bases. We now address the final issue of relating Z and H. Observe that if we take $\eth$ of (A.20), use (A.18), and eliminate the null tetrad via (A.19)–(A.22), we obtain an equation solely involving Z (and derivatives) and H. (In a similar form the conjugate equation is obtained.) This equation is explicitly given in the next appendix.

Appendix B

We present here our final set of equations in full detail. They are given in terms of Z, $H^{(\pm)}$ and $h^{(\pm)}$ bearing in mind that the latter is expressed completely in terms of $H^{(\pm)}$.

First recall that

$$H_{ij} = h_{ij}^{(+)} + H_{ij}^{(-)}, \tag{B.1}$$

$$\overline{H}_{ij} = H_{ij}^{(+)} + h_{ij}^{(-)}, \tag{B.2}$$

or explicitly,

$$h_{ij}^{(+)} = \begin{pmatrix} 0 & \frac{1}{2}(H_{01} + H_{+-}) & 0 & H_{0-} \\ -\frac{1}{2}(H_{01} + H_{+-}) & 0 & H_{1+} & 0 \\ 0 & -H_{1+} & 0 & \frac{1}{2}(H_{01} + H_{+-}) \\ -H_{0-} & 0 & -\frac{1}{2}(H_{01} + H_{+-}) & 0 \end{pmatrix},$$

$$H_{ij}^{(-)} = \begin{pmatrix} 0 & \frac{1}{2}(H_{01} - H_{+-}) & H_{0+} & 0 \\ -\frac{1}{2}(H_{01} - H_{+-}) & 0 & 0 & H_{1-} \\ -H_{0+} & 0 & 0 & -\frac{1}{2}(H_{01} - H_{+-}) \\ 0 & -H_{1-} & \frac{1}{2}(H_{01} - H_{+-}) & 0 \end{pmatrix},$$

and their conjugates.

Since the symbols $(+)$ and $(-)$ represent self-dual and anti-self-dual indices in the internal space, several components of $(H^{(-)}, h^{(-)}, H^{(+)}, h^{(+)})$ are equal to zero. Using a null basis in the internal (i,j) space and denoting the range of i,j by $(0,1,+,-)$, the non-trivial components are given by

$$(H_{01}^{(-)}, H_{0+}^{(-)}, H_{1-}^{(-)}), \qquad (h_{01}^{(-)}, h_{0+}^{(-)}, h_{1-}^{(-)}), \tag{B.3}$$

$$(H_{01}^{(+)}, H_{0-}^{(+)}, H_{1+}^{(+)}), \qquad (h_{01}^{(+)}, h_{0-}^{(+)}, h_{1+}^{(+)}), \tag{B.4}$$

with $h_{+-}^{(+)} = h_{01}^{(+)}$ and $H_{+-}^{(-)} = -H_{01}^{(-)}$. Note that, when the internal indices are raised or lowered, the 0 goes into 1, and the + into − with a change in sign. Similar changes exist for the other two components.

The explicit form of the self-dual SO(3,1) equations (A.15) for H in terms of those components are:

$$\bar{\eth}H_{01}^{(-)} - \eth h_{01}^{(-)} + H_{0+}^{(-)}h_{1-}^{(-)} - h_{0+}^{(-)}H_{1-}^{(-)} + H_{0+}^{(-)} - h_{1-}^{(-)} - \dot{\bar{\sigma}}_{\mathrm{B}}H_{1-}^{(-)} = 0, \tag{B.5}$$

$$\bar{\eth}H_{1-}^{(-)} - \eth h_{1-}^{(-)} - 2H_{01}^{(-)}h_{1-}^{(-)} + 2h_{01}^{(-)}H_{1-}^{(-)} - 2H_{01}^{(-)} = 0, \tag{B.6}$$

$$\bar{\eth}H_{0+}^{(-)} - \eth h_{0+}^{(-)} - 2H_{0+}^{(-)}h_{01}^{(-)} + 2h_{0+}^{(-)}H_{01}^{(-)} + 2h_{01}^{(-)} + 2\dot{\bar{\sigma}}_{\mathrm{B}}H_{01}^{(-)} = \eth\dot{\bar{\sigma}}_{\mathrm{B}}, \tag{B.7}$$

and

$$\eth H_{01}^{(+)} - \bar{\eth}h_{01}^{(+)} + H_{0-}^{(+)}h_{1+}^{(+)} - h_{0-}^{(+)}H_{1+}^{(+)} + H_{0-}^{(+)} - h_{1+}^{(+)} - \dot{\sigma}_{\mathrm{B}}H_{1+}^{(+)} = 0, \tag{B.8}$$

$$\eth H_{1+}^{(+)} - \bar{\eth}h_{1+}^{(+)} - 2H_{01}^{(+)}h_{1+}^{(+)} + 2h_{01}^{(+)}H_{1+}^{(+)} - 2H_{01}^{(+)} = 0, \tag{B.9}$$

$$\eth H_{0-}^{(+)} - \bar{\eth}h_{0-}^{(+)} - 2H_{0-}^{(+)}h_{01}^{(+)} + 2h_{0-}^{(+)}H_{01}^{(+)} + 2h_{01}^{(+)} + 2\dot{\sigma}_{\mathrm{B}}H_{01}^{(+)} = \bar{\eth}\dot{\sigma}_{\mathrm{B}}, \tag{B.10}$$

where we have used eqs. (A.16) and (A.17), the explicit form of A.

The equations relating H and Z are

$$\begin{aligned}\eth^2 Z_{,a} = {} & 2H_{1-}^{(-)}[h_{1+}^{(+)} + 1]n_a \\ & + [\eth h_{1+}^{(+)} - 2H_{01}^{(-)} - 2h_{1+}^{(+)}H_{01}^{(-)}]m_a + [\eth H_{1-}^{(-)} - 2H_{1-}^{(-)}h_{01}^{(+)}]\bar{m}_a \\ & + [(\dot{\sigma}_{\mathrm{B}} + h_{0-}^{(+)})(1 + h_{1+}^{(+)}) - H_{1-}^{(-)} - \eth h_{01}^{(+)} - \eth H_{01}^{(-)} \\ & + (h_{01}^{(+)} + H_{01}^{(-)})^2 + H_{1-}^{(-)}H_{0+}^{(-)}]\ell_a,\end{aligned} \tag{B.11}$$

with

$$\ell_a = Z_{,a}, \tag{B.12}$$

$$m_a = \eth Z_{,a} - H^0{}_0 Z_{,a} - H^0{}_+ m_a - H^0{}_- \bar{m}_a, \tag{B.13}$$

$$\bar{m}_a = \bar{\eth} Z_{,a} - \overline{H}^0{}_0 Z_{,a} - \overline{H}^0{}_+ m_a - \overline{H}^0{}_- \bar{m}_a, \tag{B.14}$$

$$n_a = \eth\bar{\eth} Z_{,a} + Z_{,a} - (\overline{H}^+{}_i + \bar{\eth}H^0{}_i + H^0{}_k(\overline{H} - \overline{A})^k{}_i)e^i_a. \tag{B.15}$$

These last equations are simply eqs. (A.19)–(A.22), which have been reorganized in a fashion more suitable for a perturbation expansion.

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